Appendix week 10

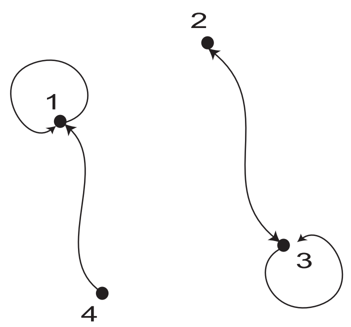
- Directed Graphs
- Classes are either identical or distinct
- Example of Relations on $\mathbb{R}^2$.
- Equivalence relations and Partitions

Directed Graphs

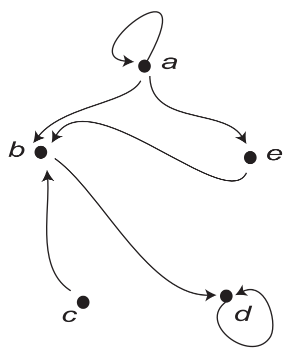
The following section will not be covered in the lectures but it might give you a better understanding of relations.

Given $\mathcal{R} \subseteq A \times A$ we can denote it by a ***directed graph*** or ***digraph*** which consists of a set of ***vertices*** (or ***nodes***) corresponding to elements of A, and ***edges*** (or ***arcs***) that connect vertices v and w if, and only if, $(v, w) \in \mathcal{R}$ with an arrow pointing from v to w.

Example If $A = \{1, 2, 3, 4\}$ and $\mathcal{R} = \{(1, 1), (3, 2), (2, 3), (4, 1), (3, 3)\}$ this relation can be drawn as



Example Starting with the digraph



we see that the relation on $\{a, b, c, d, e\}$ is

$$\{(a, a), (a, b), (a, e), (b, d), (c, b), (d, d), (e, b)\}.$$

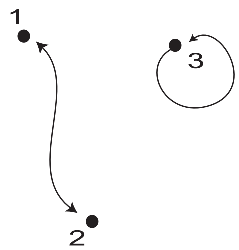
For $\mathcal{R}$ to be reflexive means that in the digraph there is a loop on every vertex.

For $\mathcal{R}$ to be symmetric means that, in the digraph, on every path between *different* vertices there will be *two* arrows.

For $\mathcal{R}$ to be transitive you have to look at every example in the digraph of a path linking three vertices using two line. Then you have to check that there is one line linking the end points (i.e. you have to check that if you can go the 'long way round' then you can go the 'direct' way. Note that in the definition of transitive the vertices x, y and z need not be different.

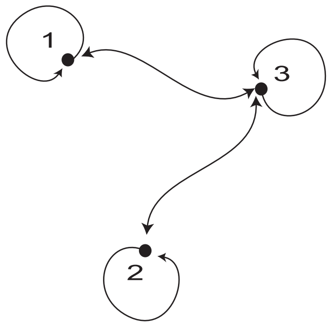
Example Let $A = \{1, 2, 3\}$.

(a) Let $\mathcal{R}_1 = \{(1, 2), (2, 1), (3, 3)\}$.



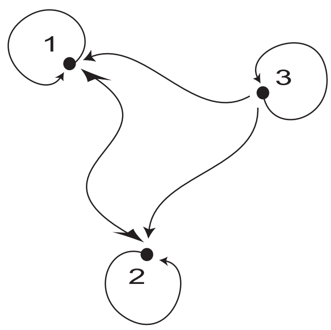
This relation is **not** reflexive, (there is no loop on 1, say) **is** symmetric, is **not** transitive $((1, 2), (2, 1) \in \mathcal{R}_1$ but $(1, 1) \notin \mathcal{R}_1)$.

(b) Let $\mathcal{R}_2 = \{(1,3), (3,1), (2,3), (3,2), (1,1), (2,2), (3,3)\}$.



This relation **is** reflexive, **is** symmetric, is **not** transitive. $((1,3), (3,2) \in \mathcal{R}_2$ but $(1,2) \notin \mathcal{R}_2)$

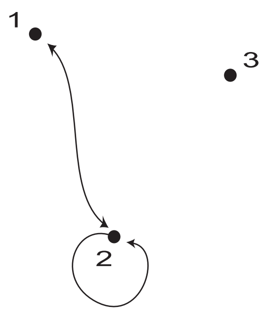
(c) Let $\mathcal{R}_3 = \{(1,2), (2,1), (3,1), (3,2), (1,1), (2,2), (3,3)\}$.



This relation **is** reflexive, is **not** symmetric $((3,2) \in \mathcal{R}_3$ but $(2,3) \notin \mathcal{R}_3)$, **is** transitive.

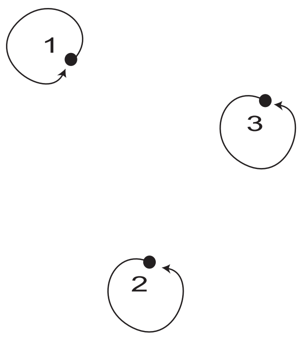
You have to be careful when checking transitivity.

(d) Let $\mathcal{R}_4 = \{(1,2), (2,1), (2,2)\}$.



This relation is **not** reflexive, **is** symmetric, and is **not** transitive since $1R2$ but $2R1$ but $1NR1$.

(e) Let $\mathcal{R}_5 = \{(1,1),(2,2),(3,3)\}$.



This relation **is** reflexive, **is** symmetric, **is** transitive.

Classes are either identical or distinct

In the lectures the following result is stated.

Theorem 1 *Suppose that $\sim$ is an equivalence relation on a non-empty set X. Then for $a, b \in X$,*

i) If $a \sim b$ then $[a] = [b]$,

ii) If $a \nsim b$ then $[a] \cap [b] = \varnothing$.

Assuming this result we can state a result that appears stronger, but is not.

Theorem 2 *Suppose that $\sim$ is an equivalence relation on a non-empty set X. Then for $a, b \in X$,*

i) $a \sim b$ iff $[a] = [b]$,

ii) $a \nsim b$ iff $[a] \cap [b] = \varnothing$.

The results now, instead of being implications, are equivalences.

Proof Because of Theorem 1 it suffice to prove

i') if $[a] = [b]$ then $a \sim b$

ii') if $[a] \cap [b] = \varnothing$ then $a \nsim b$.

i') **Assume** $[a] = [b]$**.** Proof by contradiction, so assume $a \nsim b$. Then by Theorem 1ii) we have $[a] \cap [b] = \varnothing$, in particular $[a] \neq [b]$. This contradicts the initial assumption $[a] = [b]$. Hence the assumption that $a \nsim b$ is false and thus $a \sim b$ as required.

ii') **Assume** $[a] \cap [b] = \varnothing$**.** Proof by contradiction, so assume $a \sim b$. Then by Theorem 1ii) we have $[a] = [b]$, in which case $[a] \cap [b] = [a] \neq \varnothing$. This contradicts the initial assumption $[a] \cap [b] = \varnothing$. Hence the assumption that $a \sim b$ is false and thus $a \nsim b$ as required. ■

Examples of Relations on $\mathbb{R}^2$.

Now we get some confusing examples of relations on the points of $\mathbb{R}^2$. Confusing because now (x, y) represents a point of $\mathbb{R}^2$, **not** that x and y are related.

Example 3 *Let* $X = \mathbb{R}^2$ *and* $\sim$ *be given by* $(x_1, y_1) \sim (x_2, y_2)$ *if, and only if,* $x_1 - x_2 = y_1 - y_2$. *Show that this is an equivalence relation.*

Solution left to students.

Example 4 *Let* $X = \mathbb{R}^2 \setminus \{(0, 0)\}$ *and* $\sim$ *be given by* $(x_1, y_1) \sim (x_2, y_2)$ *if, and only if,* $x_1 y_2 = x_2 y_1$. *Show that this is an equivalence relation.*

Solution The difficult property to verify is that it is transitive. Assume that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. From the definition of $\sim$ this means that

$$ad = bc \quad \text{and} \quad cf = de. \tag{1}$$

Case 1, assume $a \neq 0$. Multiply the second equation by a to get

$$acf = ade = bce, \tag{2}$$

having used the first equation. If c were equal to 0 then since $(c, d) \neq (0, 0)$ we have $d \neq 0$. Then from $ad = bc = b \times 0 = 0$ we deduce $a = 0$, which contradicts the assumption that $a \neq 0$, so we must have $c \neq 0$. Thus we can divide through (2) by this non-zero c to get $af = be$, which is the definition of $(a, b) \sim (e, f)$.

Case 2, assume $a = 0$. Since $(a, b) \neq (0, 0)$ we have $b \neq 0$. Multiply the second equation of (1) by b to get

$$bde = bcf = adf, \tag{3}$$

having used the first equation. If d were equal to 0 then since $(c, d) \neq (0, 0)$ we have $c \neq 0$. Then from $bc = ad = a \times 0 = 0$ we deduce $b = 0$, which contradicts the fact that $b \neq 0$, so $b \neq 0$. Thus we can divide through (3) by this non-zero d to get $be = af$, which is the definition of $(a, b) \sim (e, f)$.

So in all cases we get $(a, b) \sim (e, f)$, showing that the relation is transitive. ■

Example 5 *Let $X = \mathbb{R}^2$ and $\sim$ be given by $(x_1, y_1) \sim (x_2, y_2)$ if, and only if, $x_1 - x_2 = y_1 - y_2$. What do the equivalence classes look like?*

Solution To get started I would suggest choosing a point from $\mathbb{R}^2$ and calculating its equivalence class. For instance, $(3, 5)$. Then the class labeled by $(3, 5)$ is

$$\begin{aligned} [(3,5)] &= \{(x, y) : (x, y) \sim (3, 5)\} \\ &= \{(x, y) : x - 3 = y - 5\} \\ &= \{(x, y) : y = x + 2\}. \end{aligned}$$

This is the graph of the straight line $y = x + 2$, gradient 1 going through the point $(3, 5)$. In general

$$[(a, b)] = \{(x, y) : y = x + b - a\}$$

is the graph of the straight line, gradient 1, going through the point (a, b). Thus $\mathbb{R}^2/\sim$ is a collection of parallel lines that cover the plane. ■

Note how these straight lines *have* to be parallel. If there were not parallel, they would intersect but we know from above that equivalence classes are either identical or disjoint.

Example 6 *Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$ and $\sim$ be given by $(x_1, y_1) \sim (x_2, y_2)$ if, and only if, $x_1 y_2 = x_2 y_1$. What do the equivalence classes look like?*

Solution To get started I would suggest choosing a point from $\mathbb{R}^2$ and calculating its equivalence class. For instance, $(3, 5)$. Then the class labeled by $(3, 5)$ is

$$\begin{aligned} [(3,5)] &= \{(x, y) : (x, y) \sim (3, 5)\} \\ &= \{(x, y) : 5x = 3y\} \\ &= \{(x, y) : y = 5x/3\}. \end{aligned}$$

This is the graph of the straight line $y = 5x/3$. You will notice that it goes through the origin and the point we started with $(3, 5)$. This is true in general, the class

$$[(a, b)] = \{(x, y) : y = bx/a\}$$

can be represented by a straight line through the origin and the original point (a, b). ■

Note how these radial straight lines intersect at the origin, which has been removed since different classes have to be disjoint.

Equivalence relations and Partitions

Recall from the notes

Theorem 7 *Suppose that $\sim$ is an equivalence relation on a non-empty set X. Then $\Pi = X/\sim$, the set of equivalence classes, is a partition on X.*

Theorem 8 *Let Π be a partition of X and $\sim_\Pi$ the associated relation, so $a \sim_\Pi b$ if, and only if, there exists a set $A \in \Pi$ containing both a and b. Then $\sim_\Pi$ is an equivalence relation.*

Theorems 7 and 8 can be combined in two ways.

We can start with $\sim$, an equivalence relation. By Theorem 7 $X/\sim$ a partition. Then Theorem 8 says that $\sim_{(X/\sim)}$ is an equivalence relation. We will show below that we have 'gone round in a circle' and $\sim_{(X/\sim)} \equiv \sim$, i.e. the relations are identical.

Alternatively we can start with Π, a partition. Now applying Theorem 8 first we get that $\sim_\Pi$ is an equivalence relation. Next we apply Theorem 7 to get $X/\sim_\Pi$, a partition of X. We will show below that we have again 'gone round in a circle' and $X/\sim_\Pi = \Pi$, i.e. the partitions are identical.

Theorem 9 *If $\sim$ an equivalence relation and $\Pi = X/\sim$ then $\sim_\Pi$, is identical to $\sim$.*

Proof p.267 To show that $\sim_\Pi$, is identical to $\sim$ we need to show that $a \sim b$ if, and only if, $a \sim_\Pi b$ for all $a, b \in X$. Let $a, b \in X$ be given. Then

$$\begin{aligned} a \sim b \quad &\Leftrightarrow \quad [a] = [b] \quad \text{by the theorem above,} \\ &\Leftrightarrow \quad a, b \quad \text{lie in the same equivalence class in } \Pi = X/\sim, \\ &\Leftrightarrow \quad a \sim_\Pi b \quad \text{by the definition of } \sim_\Pi . \end{aligned}$$

Hence $\sim$ and $\sim_\Pi$ are the same. ■

Theorem 10 *Assume that* Π *is a partition of* X*, that* $\sim_\Pi$ *is the relation induced by* Π*, and* $X/\sim_\Pi$ *is the partition induced by* $\sim_\Pi$*. Then* $X/\sim_\Pi = \Pi$.

Proof Recall that $X/\sim_\Pi$ and Π are sets of sets. So to prove equality we need show both set inclusions $X/\sim_\Pi \subseteq \Pi$ and $\Pi \subseteq X/\sim_\Pi$.

Proof of $X/\sim_\Pi \subseteq \Pi$. Let $A \in X/\sim_\Pi$. This is a non-empty set so choose any $a \in A$. Since Π is a partition of X there exists $A' \in \Pi$ such that $a \in A'$. We need show that $A = A'$. For this we again are required to prove two set inclusions, $A \subseteq A'$ and $A' \subseteq A$.

> **Proof of** $A \subseteq A'$. Let $b \in A$, so we have a and $b \in A$. By the definition of $X/\sim_\Pi$ this means that $a \sim_\Pi b$, which in turn means that a and b are in the same part of Π, i.e. $a, b \in A'$. yet $b \in A'$ true for all $b \in A$ means that $A \subseteq A'$.
>
> **Proof of** $A' \subseteq A$. Let $c \in A'$, so $a, c \in A'$. Since they are in the same part of Π means that $a \sim_\Pi c$ which means that a and c are in the same part of $X/\sim_\Pi$, which must be A. Yet $c \in A$ for all $c \in A'$ means that $A' \subseteq A$.

Combine the two set inclusions to deduce $A = A' \in \Pi$.

True for all $A \in X/\sim_\Pi$ means that $X/\sim_\Pi \subseteq \Pi$.

The proof of $\Pi \subseteq X/\sim_\Pi$ is identical, simply replace Π by $X/\sim_\Pi$ and vice-verse throughout the proof just given. ■